ME 4555 - Lecture 18 - First order systems

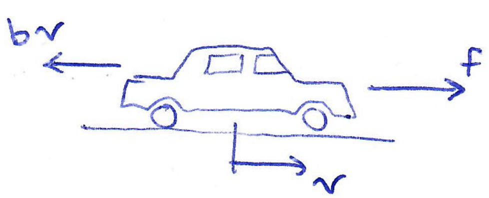
A first-order system is one that has a single pole. We saw that they can be stable, like $\frac{1}{s+1} \xrightarrow{\mathcal{L}^{-1}} e^{-t}$ or they can be unstable, like $\frac{1}{s-1} \xrightarrow{\mathcal{L}^{-1}} e^{t}$.

All stable 1^{st} order systems can be described by two numbers: Gain (K) and time constant (τ)

The general canonical form of the transfer function is:

$$\boxed{G(s) = \frac{K}{\tau s + 1}}$$

Ex. Cruise control



v is the velocity
b is the drag coefficient
f is the force

Eqn of motion: $m\dot{v} + bv = f$.

Transfer function:

$$\frac{v}{f} = \frac{1}{ms + b} = \frac{1/b}{(m/b)s + 1} \quad \text{so} \quad \begin{cases} K = 1/b \\ \tau = m/b \end{cases}$$

Step response: (input is $\frac{1}{s}$).

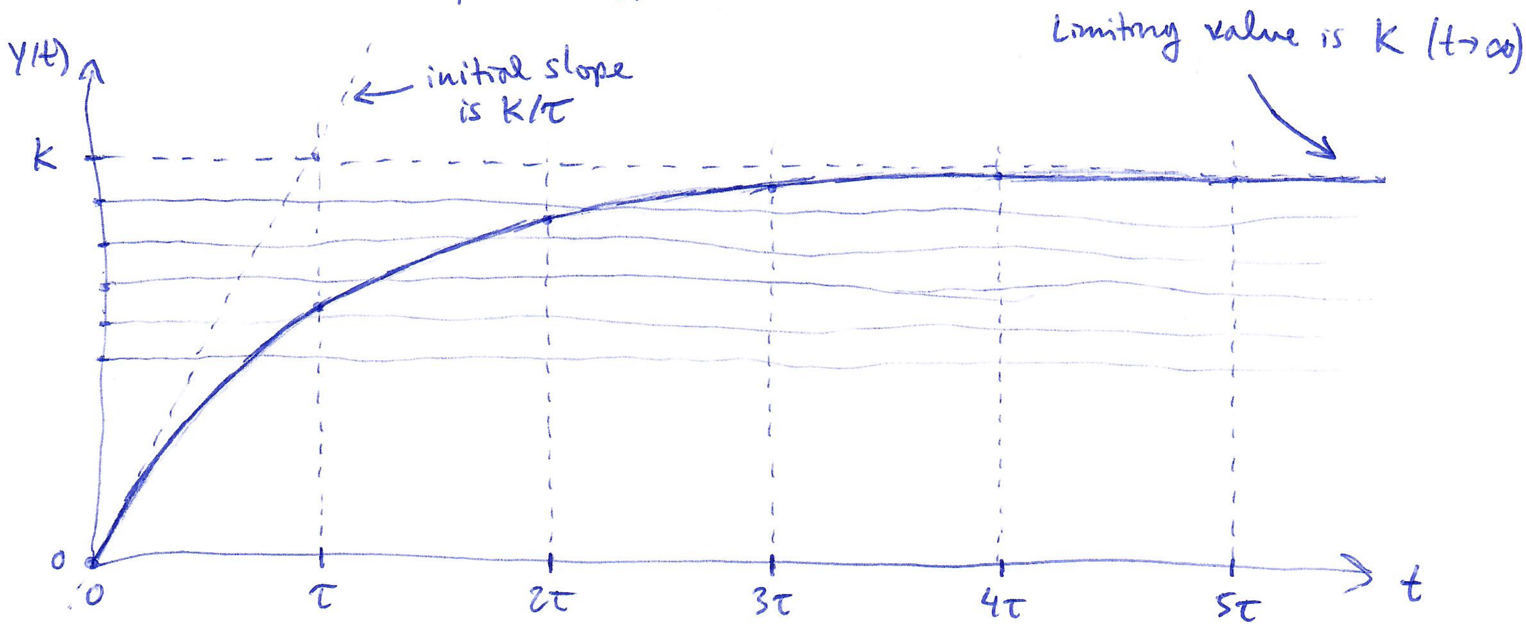
then $$Y(s) = \frac{1}{s}\cdot\frac{k}{\tau s+1} = \frac{a}{s} + \frac{b}{\tau s+1}$$

use cover-up rule: $a = k$, $b = -k\tau$.

so $$Y(s) = \frac{k}{s} - \frac{k\tau}{\tau s+1} = \frac{k}{s} - \frac{k}{s + 1/\tau}$$

$$\Rightarrow \boxed{y(t) = k\left(1 - e^{-t/\tau}\right)}$$

Here is what the plot looks like:



After time:	Fraction of final value
τ	$1-e^{-1} \approx 0.6321$
2τ	$1-e^{-2} \approx 0.8647$
3τ	$1-e^{-3} \approx 0.9502$
4τ	$1-e^{-4} \approx 0.9817$
5τ	$1-e^{-5} \approx 0.9933$

In Matlab, "rise time" is time it takes to reach 90% of final value, starting from 10%. This is about 2.2τ.

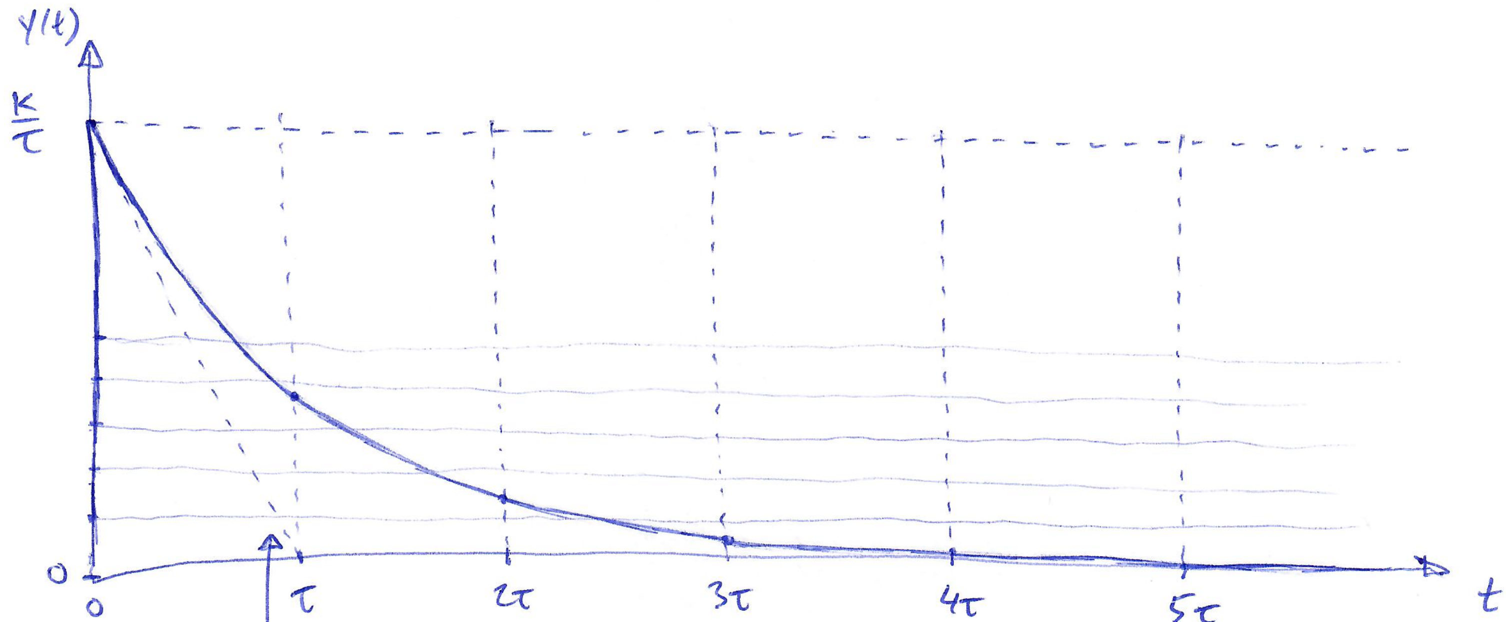
Also, "settling time" is time it takes to reach 98% of final value, which is about 3.9τ.

Impulse response (input is 1).

Then $Y(s) = \dfrac{k}{\tau s + 1} \Rightarrow y(t) = \dfrac{k}{\tau} e^{-t/\tau}$

This can be plotted in much the same way as step response.

Key difference is that max occurs at $t=0$ with value $\frac{k}{\tau}$ (not k!)

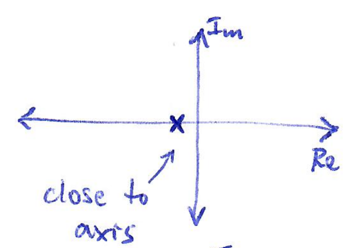


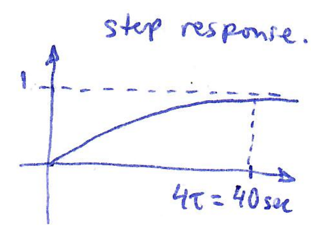
initial slope is $-\frac{k}{\tau^2}$

time constant has same interpretation as before: $4\tau \approx 98\%$ of final value, etc.

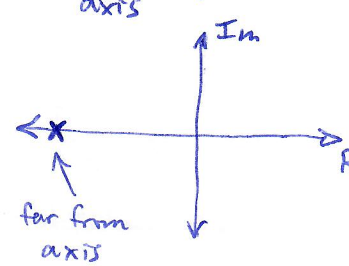
First order systems have one real pole. If stable, that pole is located in the <u>left-half</u> of the complex plane: $\boxed{\text{Pole @ } -1/\tau}$

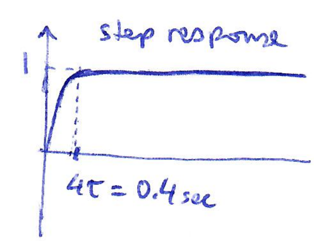
"slow pole": (at $s = -0.1$) $\quad \dfrac{1}{10s+1}$ or $\dfrac{0.1}{s+0.1}$





"fast pole": (at $s = -10$) $\quad \dfrac{1}{0.1s+1}$ or $\dfrac{10}{s+10}$





ramp response (input is $\frac{1}{s^2}$)

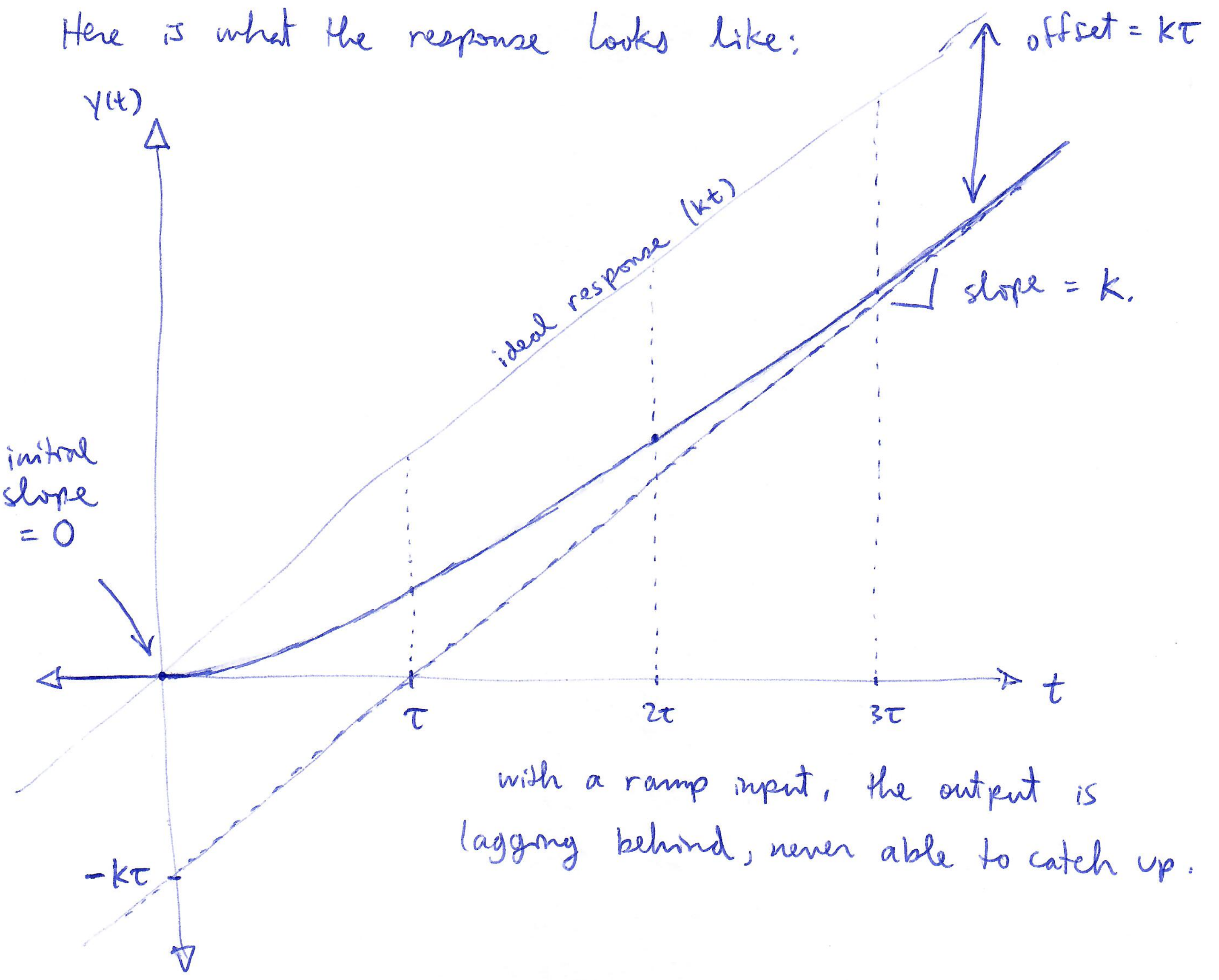
$$Y(s) = \frac{k}{s^2(\tau s+1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{\tau s + 1}$$

using modified cover-up rule: $a = -k\tau$, $b = k$, $c = k\tau^2$

so $$Y(s) = -\frac{k\tau}{s} + \frac{k}{s^2} + \frac{k\tau^2}{\tau s+1} \rightarrow \frac{k\tau}{s + 1/\tau}$$

and $$y(t) = \underbrace{-k\tau}_{\text{constant offset}} + \underbrace{kt}_{\text{ramp}} + \underbrace{k\tau e^{-t/\tau}}_{\text{decaying exponential.}}$$

Here is what the response looks like:



with a ramp input, the output is lagging behind, never able to catch up.

Ex: DC motor

Let's revisit our realistic DC motor example. The transfer function from input voltage (V_{in}) to angular speed (Ω) is:

$$\frac{\Omega}{V_{in}} = \frac{K}{(Js+b)(Ls+R)+K^2} \quad \text{where:} \begin{cases} J = 2.5\times10^{-4} & kg\cdot m^2 \\ b = 1\times10^{-4} & N\cdot m\cdot s \\ K = 0.05 & V/rad/s \\ R = 0.5 & \Omega \\ L = 1.5\times10^{-3} & H. \end{cases}$$

$$= \frac{133333}{s^2 + 333.733\,s + 6800}$$

Using the residue command in Matlab:

$[r,p,k]$ = residue([133333],[1 333.733 6800]); $\Rightarrow \begin{cases} r = \begin{bmatrix} -459.6 \\ +459.6 \end{bmatrix} \\ p = \begin{bmatrix} -311.9 \\ -21.8 \end{bmatrix} \end{cases}$

Therefore,
$$\frac{\Omega}{V_{in}} = \frac{-459.6}{s+311.9} + \frac{459.6}{s+21.8}$$

It's a sum of two first-order systems! Rewrite in canonical form:

$$\frac{\Omega}{V_{in}} = \underbrace{\frac{-1.4733}{0.0032\,s+1}}_{\text{part 1}} + \underbrace{\frac{21.08}{0.046\,s+1}}_{\text{part 2}}$$

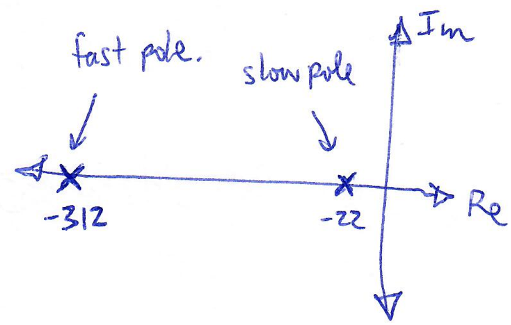
Part 1: $\begin{cases} K = -1.4733 \text{ rad/s/V} = -14 \text{ rpm/V} \\ \tau = 0.0032\,s = 3.2 \text{ ms} \end{cases}$ This is a secondary pole. Small effect (small $|K|$) and a fast response (small τ).

Part 2: $\begin{cases} K = 21.08 \text{ rad/s/V} = 201 \text{ rpm/V} \\ \tau = 0.046\,s = 46 \text{ ms.} \end{cases}$ This is the dominant pole. A large effect (large $|K|$) and a slow response (large τ)

So a step response will look essentially the same as if the first part was not present.

Ex: DC motor (cont'd).

In the complex plane, we can plot the two poles:



It turns out the fast pole is due mostly to the electrical part of the motor (R and L).

The slow pole is due to the mechanical part (J and b), which respond at a slower timescale.

If we want to make a reasonable approximation, we can set $L = 0$ (assume there is negligible inductance in the coils).

Then we get:

$$\frac{\Omega}{V_{in}} \cong \frac{K}{(Js+b)R + K^2} = \frac{400}{s + 20.4} = \frac{19.6}{0.049s + 1}$$

Notice this is $K = 19.6$ rad/s/V $= 187$ rpm/V

and $\tau = 0.049\,s = 49\,ms$

These values are close to those found for "part 2" before.

Conclusion: The presence of coil inductance causes a fast secondary pole to appear. This also moves the slow pole a bit because there is a (weak) coupling between the electrical and mechanical subsystems.